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# Rigorous Derivation and Analysis of Coupling of Kinetic Equations and Their Hydrodynamic Limits for a Simplified Boltzmann Model

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In this paper we derive rigorously the coupling of kinetic equations and their hydrodynamic limits for a simplified kinetic model. We prove the global convergence of the Chapman–Enskog expansion for this model. We then study the existence theory and asymptotic behaviour of the coupled systems. To solve the coupled problems we propose to use the transmission time marching algorithm. We then develop a convergence theory for the resulting algorithms.

**KEY WORDS:** Asymptotic behaviour; Carleman equations; coupling of kinetic equations and their hydrodynamic limits; transmission time marching algorithm.

## 1. INTRODUCTION

The Boltzmann equation is one of the most important tools in gas dynamics calculations when physical phenomena of a molecular scale cannot be neglected. In this case the model of continuum hydrodynamics cannot be any-longer considered valid for the applications. But, when the mean free path gets too small, the numerical solution of Boltzmann equations becomes impossible because the discretization step of the associated grids must be smaller than the mean free path. The classical solution consists then in replacing the Boltzmann equation by its fluid limit obtained when the mean free path goes to zero. In this paper we shall study an alternative method to these classical methods. Our analysis will be done for the linear Carleman model, which is a simplified model of the Boltzmann equation. However, to give a motivation for our methodologies, we shall present in this introduction, a brief discussion based on the full Boltzmann equation.

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The hydrodynamical limit theory aims to find the connection between the Boltzmann equation and the hydrodynamical equations (Euler and Navier–Stokes equations). This can be formulated as the search of asymptotic relationships between solutions of these equations. This connection results from two types of properties of the collision operator:

(1) Conservation properties and an entropy relation that implies that the equilibrium distribution corresponds to a Maxwellian for the limit at the order zero.

(2) The derivative of the collision operator satisfies a formal Fredholm alternative with a kernel related to the conservation properties of (1).

The compressible Euler equations are obtained formally using the conservation properties and the entropy dissipation which are consequences of the properties of the collision operator. In the Chapman–Enskog or Hilbert expansion[11] of f in $\epsilon = K_n$ the compressible Euler equations are the leading-order dynamics.

Since the compressible Euler equations generally become singular after a finite time,[22] any global in time convergence proof cannot rely on uniform regularity estimates. In ref. 1 assumptions are made on the kinetic level. The authors assume a formally consistent convergence for the fluid dynamical moments and entropy of the solution of the kinetic equations. A more detailed knowledge of the collision operator is needed in order to obtain the compressible Navier–Stokes equations. These equations arise as corrections to those of Euler at the next order in the Chapman–Enskog expansion. Strong hypothesis are needed on the regularity of solutions of the compressible Navier–Stokes equations in order to make sens of these expressions. The results available up to now consider only the case of the full space or periodic domain.[1, 2, 7, 20]

In presence of obstacles boundary conditions must be specified. When the Knudsen number is extremely small (very dense gas), the classical hypothesis of no-slip boundary conditions give accurate boundary conditions. However, for the intermediate regimes defined as the regimes for which the Knudsen number ($K_n$) which measures the ratio between the average time separating two successive collisions of a given particle and a characteristic time of the flow satisfies either:

(a) $K_n \leqslant 0.510^{-1}$ or

(b) $0.510^{-1} \leqslant K_n \leqslant 10$.

In the first case it is assumed that the Knudsen number is not extremely small ($K_n \ll 0.510^{-1}$), because otherwise the standard continuous model is valid, a breakdown of the aerodynamic theory in the region neighboring the obstacle (the so-called Knudsen layer) is observed and boundary conditions of slip type must be specified. The Chapman–Enskog expansion (or any other method based on the use of a finite number of moments) does not in general satisfy the kinetic boundary conditions. Therefore, this expansion is not valid in the Knudsen layer which is of width of the order of the mean free path.

The standard solution is to use analytical slip boundary conditions as described in refs. 13, 6, and 9. But the constants which are involved are hard to identify and their validity is questionable. On the other hand, the direct simulation of the kinetic problem is rapidly too expensive, because it requires one computational cell per mean free path. To overcome such difficulties many authors have recently tried to use intermediate asymptotic models such as Burnett equations.[28] However, in ref. 10 the authors have shown that the Burnett equations are in violation of the second law of thermodynamics thus explaining their long history of numerical difficulties.

On the other hand when the mean free path is roughly one thousand times smaller than the length of the obstacle these asymptotic models are no-longer valid.

In ref. 23, the author proposed a fundamental strategy that permits the coupling of different models and/or different approximations to compute the solution of the exterior domain problem. This strategy has been applied to the coupling of Boltzmann and Euler or Navier–Stokes equations.[23–27, 14–19, 3–5] Thus for the solution of intermediate regimes (as defined above) this method consists of coupling the hydrodynamics equations (Euler or Navier–Stokes equations) with the Boltzmann equation. The resulting method involves additional mathematical difficulties related to the matching of equations of the two models. However, this approach has several computational and physical advantages. One of the great advantages is the use of the correct model related to the physical features of the flow.

The application of this approach to the solution of Boltzmann equation for external domain problem consists of the following steps

(1) Domain decomposition of the external field into the domains, possibly overlapped, of validity of Boltzmann equation and of the hydrodynamic equations.

(2) Solution of the kinetic equations in the domain of validity of Boltzmann equation.

(3) Solution of hydrodynamic equations in their domain of validity.

(4) Coupling of solutions to the two models, the continuous and kinetic ones.

We should notice here that these methodologies yield coupled problems through transmission boundary conditions, which are obtained at the modelling level of the physical phenomena. Therefore these methods *cannot and should not* be classified as domain decomposition methods. We shall develop in the next sections the mathematical foundations of these methodologies for the Carleman model of Boltzmann equations. In the next section we shall derive the hydrodynamic limit of the kinetic model and then prove global convergence of Chapman–Enskog expansion. In Sections 3 and 4 we shall study these methodologies for the simplified kinetic model. This includes the existence theory, the asymptotic behavior and the convergence analysis of the transmission time marching algorithm applied to these coupled problems.

## 2. THE KINETIC EQUATIONS AND THEIR HYDRODYNAMIC LIMITS

We consider in this section the following linearized Carleman system.[8]

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=a(v-u) \qquad \text{on} \quad ]0,1[\times]0,T[, \tag{1}$$

$$\frac{\partial v}{\partial t}-\frac{\partial v}{\partial x}=a(u-v) \qquad \text{on} \quad ]0,1[\times]0,T[, \tag{2}$$

$$u(0,\cdot)=u_0 \qquad v(0,\cdot)=v_0, \tag{3}$$

$$u(t,0)=g(t) \qquad v(t,1)=h(t), \tag{4}$$

where $t\in]0,T[$, $T>0$, $x\in[0,1]$ and $u(t,x)$, $v(t,x)$ are functions of $x$ which represent probability densities for particles moving in the positive and negative $x-$direction, respectively. $a=\frac{1}{\epsilon}$ is a positive constant with $\epsilon$ the mean free path, $g(t)$ and $h(t)$ are two nonegative given functions describing the boundary data, and $u_0$ and $v_0$ are two nonegative functions describing the initial data. This model describes a random walk in one dimension. System (1)–(4) has a unique strong solution. In this section, we shall derive the hydrodynamic limit of this kinetic model and prove a result about the convergence of the kinetic model to its hydrodynamic limit as the mean free path $\epsilon$ goes to 0. We introduce the notations

$$U=\begin{bmatrix} u \\ v \end{bmatrix}, \qquad M=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad U_0=\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{5}$$

and

$$Q(U,W)=\frac{1}{2}[(u-v)+(w_1-w_2)]\begin{bmatrix}-1\\1\end{bmatrix}, \tag{6}$$

Now Eqs. (1)–(2) can be written as follows

$$\frac{\partial U}{\partial t}+M\frac{\partial U}{\partial x}=\frac{1}{\epsilon}Q(U,U) \tag{7}$$

The only collision invariant of the collision operator $Q(U,U)$ is $\psi_1=[1,1]$. The fluid moment is then defined as $\rho=\langle\psi_1,U\rangle=u+v$. The local Maxwellians correspond to the vector solutions of the equation $Q(W,W)=0$. In this case, $w_1=\frac{1}{2}\rho$, $w_2=\frac{1}{2}\rho$. Now let $\psi_2$ be such that $\psi_1$ and $\psi_2$ form a basis for $\mathbb{R}^2$: $\psi_2=[1,-1]$. Introducing $m=\langle\psi_2,U\rangle=u-v$, we obtain: $U=\rho\psi_1+m\psi_2$. Here $\rho$ corresponds to the fluid component of $U$ while $m$ is the non-hydrodynamic component. Taking the projection of Eq. (7) into $\psi_1$ and $\psi_2$, we obtain

$$\frac{\partial\rho}{\partial t}+\frac{\partial m}{\partial x}=0 \tag{8}$$

$$\frac{\partial m}{\partial t}+\frac{\partial\rho}{\partial x}=-\frac{2}{\epsilon}m \tag{9}$$

$$\rho(x,0)=u_0(x)+v_0(x)=\eta(x) \tag{10}$$

$$m(x,0)=u_0(x)-v_0(x)=\mu(x) \tag{11}$$

The Chapman–Enskog procedure consists in expanding the non-hydrodynamic component $m$ in a power series of $\epsilon$: $m=\sum_n\epsilon^n m_n$. Inserting this in the equation and comparing terms of like powers (in $\epsilon$), we obtain

$$m_0=0 \tag{12}$$

$$\frac{\partial\rho}{\partial x}=-2m_1 \tag{13}$$

$$\frac{\partial m_1}{\partial t}=-2m_2 \tag{14}$$

$$\frac{\partial m_n}{\partial t}=-2m_{n+1} \tag{15}$$

Combining the above equations and neglecting the terms in $\epsilon^2$ we get $\frac{\partial\rho}{\partial t}+\epsilon\frac{\partial m_1}{\partial x}=0$. Hence we have

$$\frac{\partial\rho}{\partial t}-\frac{1}{2}\epsilon\frac{\partial^2\rho}{\partial x^2}=0 \tag{16}$$

which corresponds to the first order approximation in the Chapman–Enskog expansion. This equation is a Navier–Stokes like approximation of the Carleman equations (1)–(4). Assuming that the solution $\rho$ of the hydrodynamic equation (16) is bounded below by a positive constant

$$\rho>\rho_0>0, \tag{17}$$

we obtain

$$\min_{\Gamma_T}\rho(x,t)\leqslant\rho(x,t)\leqslant\max_{\Gamma_T}\rho(x,t) \tag{18}$$

where $\Gamma_T=\Gamma\times[0,T]$. We shall assume that the boundary terms $\min_{\Gamma_T}\rho(x,t)$ and $\max_{\Gamma_T}\rho(x,t)$ are independent of $T$ then $\rho$ is bounded independently of $T$. This hypothesis is important for the obtention of the main theorem of this section.

Since the Chapman–Enskog terms do not satisfy the initial conditions we shall introduce correcting terms. Let $\tau=\frac{t}{\epsilon}$, substituting into Eqs. (8)–(11), we obtain

$$\frac{1}{\epsilon}\frac{\partial\rho}{\partial\tau}(\epsilon\tau)+\frac{\partial m}{\partial x}(x,\epsilon\tau)=0$$

$$\frac{1}{\epsilon}\frac{\partial m}{\partial\tau}(\epsilon\tau)+\frac{\partial\rho}{\partial x}(x,\epsilon\tau)=-\frac{2}{\epsilon}m(\epsilon\tau)$$

Let $\tilde{\rho}$ and $\tilde{m}$ be the initial layer solutions and $\rho^c$, $m^c$ denote the complete solution of Problem (8)–(11): $\rho^c=\rho+\tilde{\rho}$, $m^c=m+\tilde{m}$. We expand the initial layer solutions in power solutions of $\epsilon$: $\tilde{\rho}=\sum\epsilon^n\tilde{\rho}_n$, $\tilde{m}=\sum\epsilon^n\tilde{m}_n$. The Chapman–Enskog solutions are expanded in Taylor series around the point $\epsilon=0$.

$$\begin{aligned}\rho(x,\epsilon\tau)&=\rho(x,0)+\epsilon\tau\frac{\partial\rho}{\partial t}(x,0)+\cdots\\ m_n(x,\epsilon\tau)&=m_n(x,0)+\epsilon\tau\frac{\partial m_n}{\partial t}(x,0)+\cdots\end{aligned} \tag{19}$$

We also write

$$\begin{aligned} \rho(x,0) &= \eta_0(x) + \epsilon\eta_1(x) + \cdots \\ m_n(x,0) &= m_n^{(0)}(x) + \epsilon m_n^{(1)}(x) + \cdots \end{aligned} \tag{20}$$

We then obtain

$$\begin{aligned} \rho^c &= (\tilde{\rho}_0 + \rho(x,0)) + \epsilon\left(\tilde{\rho}_1 + \tau\frac{\partial\rho}{\partial\tau}(x,0)\right) + \cdots \\ m^c &= (\tilde{m}_0 + m_0(x,0)) + \epsilon(\tilde{m}_1 + \tau m_1(x,0) + \cdots \end{aligned} \tag{21}$$

Substituting into Eqs. (8)–(11), we obtain

$$\frac{\partial\tilde{\rho}_0}{\partial\tau} = 0, \qquad \frac{\partial\tilde{\rho}_1}{\partial\tau} + \frac{\partial\tilde{m}_0}{\partial x} = 0$$

$$\frac{\partial\tilde{m}_0}{\partial\tau} = -2\tilde{m}_0, \qquad \frac{\partial\tilde{m}_1}{\partial\tau} + \frac{\partial\tilde{\rho}_0}{\partial x} = -2\tilde{m}_1$$

Subject to the conditions $\lim_{\tau\to\infty}\tilde{\rho}_0 = 0$ and $\lim_{\tau\to\infty}\tilde{\rho}_1 = 0$, we have

$$\begin{aligned} \tilde{\rho}_0(\tau) &= 0, & \tilde{m}_0(\tau) &= \tilde{\mu}_0(x)\, e^{-2\tau} \\ \tilde{\rho}_1 &= -\frac{1}{2}\frac{\partial\tilde{\mu}_0}{\partial x}\, e^{-2\tau}, & \tilde{m}_1 &= \tilde{\mu}_1(x)\, e^{-2\tau} \end{aligned} \tag{22}$$

Expanding now $\eta$ and $\mu$ in power series, we obtain

$$\begin{aligned} \eta(x) &= (\eta_0(x) + \epsilon\eta_1(x) + \cdots) + (\tilde{\rho}_0(0) + \epsilon\tilde{\rho}_1(0) + \cdots) \\ \mu(x) &= m_0^{(0)} + \tilde{\mu}_0 + \epsilon(m_0^{(1)} + m_1^{(0)} + \tilde{\mu}_1) + \cdots \end{aligned} \tag{23}$$

Using the relations $m_0 = 0$ and $\tilde{\rho}_0 = 0$, we obtain

$$\begin{aligned} \eta(x) &= \eta_0(x) + \epsilon(\eta_1(x) + \tilde{\rho}_1(0)) + \cdots \\ \mu(x) &= \tilde{\mu}_0 + \epsilon(m_1^{(0)} + \tilde{\mu}_1) + \cdots \end{aligned} \tag{24}$$

Now we want $\eta(x) = \eta_0(x)$. So we choose $\eta_i$ such that

$$\eta_1(x) + \tilde{\rho}_1(0) = 0, \qquad \eta_i(x) = 0 \quad \forall i \geqslant 2 \tag{25}$$

Similarly we want $\mu = \tilde{\mu}_0$. So we choose: $m_1^{(0)} + \tilde{\mu}_1 = 0$. We then obtain

$$\eta_1(x) = -\frac{1}{2}\frac{\partial \tilde{\mu}_0}{\partial x} = -\frac{1}{2}\frac{\partial \mu}{\partial x}$$
$$\tilde{\mu}_1 = -m_1^{(0)} = \tilde{m}_1 = -\frac{1}{2}\frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{\partial \eta_0}{\partial x} \tag{26}$$

Since $m_1 = -\frac{1}{2}\frac{\partial \rho}{\partial x}$, we obtain $m_1(x,0) = -\frac{1}{2}\frac{\partial \rho}{\partial x}(x,0) = -\frac{1}{2}\frac{\partial \eta}{\partial x}$. Hence we have

$$\rho^c(x,0) = \eta(x) = \eta_0$$
$$m^c(x,0) = \tilde{\mu}_0$$
$$\tilde{\rho}_0(\tau) = 0$$
$$\tilde{m}_0(\tau) = \tilde{\mu}_0(x)\, e^{-2\tau}$$
$$\tilde{\rho}_1(\tau) = -\frac{1}{2}\frac{\partial \tilde{\mu}_0}{\partial x}\, e^{-2\tau}$$
$$\tilde{m}_1 = \tilde{\mu}_1(x)\, e^{-2\tau}$$

Now assume that: $\rho^c = \rho + \epsilon\tilde{\rho}_1 + \epsilon^2 u$, $m^c = \tilde{m}_0 + \epsilon m_1 + \epsilon^2 v$, satisfy the equations

$$\frac{\partial \rho^c}{\partial t} + \frac{\partial m^c}{\partial x} = 0$$
$$\frac{\partial m^c}{\partial t} + \frac{\partial \rho^c}{\partial x} = -\frac{2}{\epsilon} m^c$$

Using the equation $\frac{\partial \rho}{\partial t} + \epsilon\frac{\partial m_1}{\partial x} = 0$, we obtain

$$\epsilon\frac{\partial \tilde{\rho}_1}{\partial t} + \epsilon^2\frac{\partial u}{\partial t} + \frac{\partial \tilde{m}_0}{\partial x} + \epsilon\frac{\partial \tilde{m}_1}{\partial t} + \epsilon^2\frac{\partial v}{\partial x} = 0$$
$$\frac{\partial \tilde{\rho}_1}{\partial \tau} + \frac{\partial \tilde{m}_0}{\partial x} + \epsilon\frac{\partial \tilde{m}_1}{\partial t} + \epsilon^2\left(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x}\right) = 0$$

This yields

$$\frac{\partial u}{\partial t}+\frac{\partial v}{\partial x}=-\frac{1}{\epsilon}\frac{\partial \tilde{m}_1}{\partial x} \tag{27}$$

On the other hand, we have

$$\epsilon^2\frac{\partial v}{\partial t}+\epsilon^2\frac{\partial u}{\partial x}=-\frac{\partial \tilde{m}_0}{\partial t}-\epsilon\frac{\partial m_1}{\partial t}-\epsilon\frac{\partial \tilde{m}_1}{\partial t}-\frac{\partial \rho}{\partial x}-\epsilon\frac{\partial \tilde{\rho}_1}{\partial x}-\frac{2}{\epsilon}\tilde{m}_0-2m_1-2\tilde{m}_1-2\epsilon v \tag{28}$$

Since $\frac{\partial \tilde{m}_0}{\partial t}=\frac{1}{\epsilon}\frac{\partial \tilde{m}_0}{\partial \tau}=-\frac{2}{\epsilon}\tilde{m}_0$ and $m_1=-\frac{1}{2}\frac{\partial \rho}{\partial x}$, we obtain

$$\begin{aligned}\frac{\partial v}{\partial t}+\frac{\partial u}{\partial x}&=-\frac{2}{\epsilon}v-\frac{2}{\epsilon^2}\tilde{m}_1-\frac{1}{\epsilon}\frac{\partial m_1}{\partial t}-\frac{1}{\epsilon}\frac{\partial \tilde{m}_1}{\partial t}-\frac{1}{\epsilon}\frac{\partial \tilde{\rho}_1}{\partial x}\\&=-\frac{2}{\epsilon}v-\frac{1}{\epsilon}\frac{\partial m_1}{\partial t}-\frac{1}{\epsilon}\frac{\partial \tilde{\rho}_1}{\partial x}\\&=-\frac{2}{\epsilon}v-\frac{1}{\epsilon}W(x,t)\end{aligned} \tag{29}$$

Therefore $u$ and $v$ satisfy the following system of equations

$$\frac{\partial u}{\partial t}+\frac{\partial v}{\partial x}=-\frac{1}{\epsilon}\frac{\partial \tilde{m}_1}{\partial x}$$

$$\frac{\partial v}{\partial t}+\frac{\partial u}{\partial x}=-\frac{2}{\epsilon}v-\frac{1}{\epsilon}W(x,t)$$

We now derive the boundary conditions for the error equations

$$\rho(x,0)=\eta(x)+\epsilon\eta_1 m_1=-\frac{1}{2}\frac{\partial \rho}{\partial x}$$

$$m_1(x,0)=-\frac{1}{2}\frac{\partial \eta}{\partial x}-\frac{1}{2}\epsilon\frac{\partial \eta_1}{\partial x}\frac{\partial m_1}{\partial \epsilon}=-\frac{1}{2}\frac{\partial \eta_1}{\partial x}$$

Since $m^c(x,0)=\tilde{\mu}_0=\mu$, we have

$$\tilde{\mu}_1=-m_1^{(1)} \qquad \tilde{\mu}_2=-m_1^{(1)} \quad (\text{term in } \epsilon^2)$$

This yields the initial data for the error equations

$$u(x,0)=0, \qquad v(x,0)=-m_1^{(1)}(\theta)$$

where

$$m_1^{(1)}(\theta)=\frac{\partial m_1}{\partial \epsilon}(\theta\epsilon)$$

is the remainder in the Taylor expansion of $m_1(x,0)$ around the point $\epsilon=0$. The function $m_1^{(1)}(\theta)$ is bounded and continuous. Now set $w=\frac{1}{2}u\psi_1+\frac{1}{2}v\psi_2$ then Eqs. (8)–(11) become

$$\frac{\partial w_1}{\partial t}+\frac{\partial w_1}{\partial x}=\frac{1}{2}\left(-\frac{1}{\epsilon}\frac{\partial \tilde{m}_1}{\partial x}-\frac{2}{\epsilon}v-\frac{1}{\epsilon}W\right)$$

$$\frac{\partial w_1}{\partial t}-\frac{\partial w_1}{\partial x}=\frac{1}{2}\left(-\frac{1}{\epsilon}\frac{\partial \tilde{m}_1}{\partial x}+\frac{2}{\epsilon}v+\frac{1}{\epsilon}W\right)$$

Hence we have

$$\begin{aligned}\frac{\partial w}{\partial t}+V\frac{\partial w}{\partial x}&=-\frac{1}{2\epsilon}\left(\frac{\partial \tilde{m}_1}{\partial x}\psi_1+W\psi_2\right)-\frac{1}{\epsilon}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}w\\ &=-\frac{1}{2\epsilon}\left(\frac{\partial \tilde{m}_1}{\partial x}\psi_1+W\psi_2\right)-\frac{1}{\epsilon}Lw \end{aligned} \tag{30}$$

$$w(x,0)=\frac{1}{2}v(x,0)\,\psi_2=-\frac{1}{2}m_1^{(1)}(\theta)\,\psi_2=w_0 \tag{31}$$

Next we give some properties of $W$ and $\frac{\partial \tilde{m}_1}{\partial x}$. This is stated in the following lemma.

**Lemma 2.1.** The functions $W$ and $\frac{\partial \tilde{m}_1}{\partial x}$ satisfy

$$\int_0^\infty ||W||\,dt<C\epsilon \tag{32}$$

$$\int_0^\infty \left\|\frac{\partial \tilde{m}_1}{\partial x}\right\|dt<C\epsilon \tag{33}$$

where $||\cdot||$ denotes the norm in $C([0,1])$ and $C$ is a constant independent of $\epsilon$.

*Proof.* Using the relation (13) it is enough to prove that $\rho$ and its derivatives satisfy (32). Without loss of generality we may assume that $\rho(0)=\rho(1)=0$. Let $\varphi$ be a positive function to be precised later. Multiplying Eq. (16) by $\varphi\rho$ and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int\varphi\rho^2+\frac{1}{2}\epsilon\int\varphi\left|\frac{\partial\rho}{\partial x}\right|^2-\frac{\epsilon}{4}\int\varphi''\rho^2=0 \tag{34}$$

We then have

$$\frac{1}{2}\frac{d}{dt}\int\varphi\rho^2-\frac{\epsilon}{4}\int\varphi''\rho^2\leqslant 0 \tag{35}$$

We shall construct $\varphi$ such that $-\frac{\epsilon}{4}\varphi''=\gamma\varphi$, where $\gamma>0$ is a constant to be precised later. Setting $\psi=\int\varphi\rho^2$, we obtain

$$\frac{d}{dt}\psi(t)+2\gamma\psi(t)\leqslant 0 \tag{36}$$

from which we deduce that

$$\psi\leqslant e^{-2\gamma t}\psi(0) \tag{37}$$

On the other hand using (17), we obtain

$$\psi(t)\geqslant\int\varphi\rho_0 \tag{38}$$

We can choose $\varphi>0$ such that $\gamma=\frac{1}{\epsilon}$, $\varphi\geqslant\frac{\alpha}{\rho_0^2}\,||\rho||$ with $\alpha>0$ independent of $\epsilon$, and such that we have

$$\psi(t)\geqslant\alpha\,||\rho|| \tag{39}$$

and

$$\int_0^\infty||\rho||<C\epsilon.$$

Similarly we prove that

$$\int_0^\infty\left\|\frac{\partial\rho}{\partial x}\right\|<C\epsilon,\qquad\int_0^\infty\left\|\frac{\partial\rho}{\partial t}\right\|<C\epsilon,\qquad\text{etc}\cdots$$

From this we conclude that $\int_0^\infty \|W\| \, dt < \infty$. The proof of the second inequality in the lemma is a consequence of the construction of $\tilde{\mu}_0$ and $\tilde{\mu}_1$. The lemma is proved.

We shall now study the problem (30)–(31). This problem has a mild solution in the Banach space $B = C[0, 1] \times C[0, 1]$. In fact, the operator $A_\epsilon w = V \frac{\partial w}{\partial x} + \frac{1}{\epsilon} Lw$ for $w \in D(A_\epsilon)$ where

$$D(A_\epsilon) = \{ f \in B;\ f \in C^1 \}$$

has a closure $\bar{A}_\epsilon$ that generates in $B$ a two-parameter family of contractions $V(t, s)$. More precisely we have the lemma

**Lemma 2.2.** The closure $\bar{A}_\epsilon$ generates in $B$ a two-parameter family of contractions $V(t, s)$.

The proof of this lemma is similar to the proof of a similar lemma stated in ref. 21. The operators $V(t, s)$ allow us to write Problem (30)–(31) in the integral form

$$w(t) = V(t, 0)\, w_0 - \frac{1}{2\epsilon} \int_0^t V(t, s) \left( \frac{\partial \tilde{m}_1}{\partial x} \psi_1 + W \psi_2 \right) ds \tag{40}$$

We have the following lemma.

**Lemma 2.3.** The problem (40) has a unique solution $w(t)$. Moreover $w$ satisfies

$$\|w(t)\| \leqslant C$$

where $C$ is a constant independent of $\epsilon$ and $T$.

***Proof.*** If $w$ is a solution of Problem (40) then using Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} \|w(t)\| &\leqslant C + \frac{1}{2\epsilon} \int_0^t |||V(t, s)||| \left( \left\| \frac{\partial \tilde{m}_1}{\partial x} \right\| + \|W\| \right) \\ &\leqslant C + \frac{1}{2\epsilon} \int_0^\infty |||V(t, s)||| \left( \left\| \frac{\partial \tilde{m}_1}{\partial x} \right\| + \|W\| \right) \\ &\leqslant C \end{aligned}$$

The existence and uniqueness of a solution to Problem (40) is a consequence of the theory of Volterra integral equations.

Combining the last lemmas together with the construction of the Chapman–Enskog and the initial layer expansions we obtain the following theorem.

**Theorem 2.1.** The initial problem (30)–(31) has a mild solution $f(t)$ on the time interval $[0, +\infty[$. The solution $f(t)$ is uniformly bounded in $B$ for $t \in [0, +\infty[$. Moreover, we have the following estimate

$$\|f(t) - \tfrac{1}{2}(\rho + \epsilon \tilde{\rho}_1)\, \psi_1 - \tfrac{1}{2}(\tilde{m}_0 + \epsilon m_1 + \epsilon \tilde{m}_1)\, \psi_2\| \leqslant c\epsilon^2,$$

where

$$\tfrac{1}{2}\, \rho \psi_1 + \tfrac{1}{2}\, \epsilon m_1 \psi_2$$

and

$$\tfrac{1}{2}\, \epsilon \tilde{\rho}_1 \psi_1 + \tfrac{1}{2}(\tilde{m}_0 + \epsilon \tilde{m}_1)\, \psi_2$$

are respectively the Chapman–Enskog expansion and the initial layer expansion obtained at the beginning of this section.

This result differs from the result in ref. 21 in that it provides global convergence of the Chapman–Enskog expansion. However, our results are obtained for a linear model while the results in ref. 21 are obtained for a nonlinear model.

**Remark 2.1.** In the next sections we shall consider initial data which are of the form of a local Maxwellian at $t = 0$. In this case the $\epsilon$-dependence disappears on the level of the Navier–Stokes equations and the initial layer corrections are not needed.

## 3. COUPLING OF KINETIC AND THEIR HYDRODYNAMIC LIMITS: THE MODEL (α)

Let $X = ]0, 1[$ and $X_1 = ]0, h_1[$, $X_2 = ]h_1, 1[$ $(0 < h_1 < 1)$. Assume now that in $X_1$ the hydrodynamic theory gives poor approximation to the density of particles, but gives good approximation outside $X_1$. Then we propose the following physical model consisting of two models: the kinetic model (here the simplified (Carleman) Boltzmann model) used in $X_1$ and its hydrodynamic approximation used in $X_2$,

$$\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}=a(v_1-u_1), \qquad x\in ]0,h_1[, \quad t>0, \tag{41}$$

$$\frac{\partial v_1}{\partial t}-\frac{\partial v_1}{\partial x}=a(u_1-v_1), \qquad x\in ]0,h_1[, \quad t>0, \tag{42}$$

$$u_1(t,0)=g(t) \qquad v_1(t,h_1)=\frac{1}{2}\left(\rho_g+\frac{\epsilon}{2}\frac{\partial \rho_g}{\partial x}\right)(t,h_1), \quad t>0, \tag{43}$$

$$\frac{\partial \rho_g}{\partial t}-\frac{1}{2}\epsilon\frac{\partial^2 \rho_g}{\partial x^2}=0, \qquad x\in ]h_1,1[, \quad t>0, \tag{44}$$

$$\rho_g(h_1)=u_1(h_1)+v_1(h_1) \qquad \rho_g(\cdot,1)=h(t), \quad t>0 \tag{45}$$

$$u_1(0,\cdot)=u_{10} \qquad v_1(0,\cdot)=v_{10}, \qquad \rho_g(0,.)=\rho_{g0} \tag{46}$$

where $g, h, u_{10}, v_{10}$, and $\rho_{g0}$ are given nonegative data. The transmission boundary conditions (43) and (45) are obtained using the hydrodynamic limit analysis performed in Section 2. The resulting model is called the model ($\alpha$). Using the analysis of Section 2, it is clear that this model is rigorously justified. Notice that Problems (41)–(43) and (44)–(45) are only coupled by their boundary conditions. Therfore they can be solved by two independent solution techniques. In this section we shall develop the existence and asymptotic theory for the model ($\alpha$). We shall also propose an algorithm for the solution of this coupled problem and then establish its convergence theory.

### 3.1. Existence Theory

In this paragraph, we shall study the existence and uniqueness of a solution for the coupled problem (41)–(46). We shall work in the Hilbert space

$$H=(L^2[0,h_1])^2\times(L^2[h_1,1]),$$

with the following norm

$$\|(w_1,w_2,w_3)\|=(\|w_1\|^2_{L^2[0,h_1]}+\|w_2\|^2_{L^2[0,h_1]}+\|w_3\|^2_{L^2[h_1,1]})^{\frac{1}{2}}$$

We have the following result.

**Theorem 3.1.** Assume that $(u_{10}, v_{10}, \rho_{g0}) \in H$, then the coupled problem (41)–(46) has a unique strong solution $(u_1, v_1, \rho_g)$.

We shall give the proof of this theorem for the homogeneous boundary conditions: $g(t) \equiv 0$ and $h(t) \equiv 0$. By a standard argument the proof in the nonhomogeneous case can be reduced to the homogeneous case.

We introduce an operator $A$ on $H$ as follows

$$A(w_1, w_2, w_3) = \begin{pmatrix} w_1' + a(w_1 - w_2) \\ -w_2' + a(w_2 - w_1) \\ -\frac{1}{2}\epsilon w_3'' \end{pmatrix}, \tag{47}$$

$$D(A) = \left\{ \begin{array}{r} (w_1, w_2, w_3) \in H \mid w_1',\ w_2' \in L^2[0, h_1],\ \text{and}\, w_3'' \in L^2[h_1, 1] \\ w_1(0) = 0,\ w_2(h_1) = \frac{1}{2}w_3(h_1) + \frac{\epsilon}{4}w_3'(h_1),\ w_3(h_1) = w_1(h_1) + w_2(h_1), \\ \text{and}\ w_3(1) = 0 \end{array} \right\}.$$

It is clear that $D(A)$ is dense in $H$. We shall apply the Hille–Yosida theorem. Let $\lambda$ be a real number and let $f \in X$. We shall study the problem

$$\text{find}\ w \in D(A)\ \text{solution of}\ Aw + \lambda w = f, \tag{48}$$

which corresponds to finding $w \in D(A)$ such that

$$w_1' + (a + \lambda)\, w_1 - a w_2 = f_1, \tag{49}$$

$$-w_2' + (a + \lambda)\, w_2 - a w_1 = f_2, \tag{50}$$

$$-\tfrac{1}{2}\epsilon w_3'' + \lambda w_3 = f_3, \tag{51}$$

By a density argument we may assume that $f_1$, $f_2$, and $f_3$ are continuous. By elementary methods we obtain the general solution of System (49)–(51).

We shall now prove the estimate: $||w||_\varphi \leqslant \frac{1}{\lambda - 2} ||f||_\varphi \ \forall \lambda > 2$, where $||\cdot||_\varphi$ is a norm equivalent to the norm $||\cdot||$ to be precised later. Let $\varphi_1$, $\varphi_2$, and $\varphi_3$ be three positive functions (of $x$ only) to be precised later. Multiplying Eqs. (49)–(51) respectively by $\varphi_1 w_1$, $\varphi_2 w_2$, and $\varphi_3 w_3$, integrating by parts, using Cauchy–Schwarz inequality, and combining the resulting equations, we obtain

$$\int\left[\frac{a+\lambda}{2}\varphi_1-\frac{1}{2}\varphi_1'\right]w_1^2+\frac{1}{2}(\varphi_1 w_1^2)_0^{h1}$$
$$-\frac{a}{2}\int\varphi_1 w_2^2+\int\left[\frac{a+\lambda}{2}\varphi_2+\frac{1}{2}\varphi_2'\right]w_2^2-\frac{1}{2}(\varphi_2 w_2^2)_0^{h_1}$$
$$-\frac{a}{2}\int\varphi_2 w_1^2+\int\left(\frac{\lambda}{2}\varphi_3-\frac{\epsilon}{4}\varphi_3''\right)w_3^2+\frac{1}{2}\epsilon\int\varphi_3(w_3')^2$$
$$+\frac{\epsilon}{4}(\varphi_3' w_3^2)_{h_1}^1-\frac{1}{2}\epsilon(\varphi_3 w_3' w_3)_{h_1}^1$$
$$\leqslant\frac{1}{2\lambda}\int\varphi_1 f_1^2+\frac{1}{2\lambda}\int\varphi_2 f_2^2+\frac{1}{2\lambda}\int\varphi_3 f_3^2 \tag{52}$$

Using the coupling boundary conditions we obtain

$$\frac{1}{2}(\varphi_1 w_1^2)_0^{h_1}-\frac{1}{2}(\varphi_2 w_2^2)_0^{h_1}+\frac{\epsilon}{4}(\varphi_3' w_3^2)_{h_1}^1-\frac{1}{2}\epsilon(\varphi_3 w_3' w_3)_{h_1}^1$$
$$=\frac{1}{2}\Big[\varphi_1(h_1)\,w_1^2(h_1)-\varphi_2(h_1)\,w_2^2(h_1)+\varphi_2(0)\,w_2^2(0)$$
$$-\frac{\epsilon}{2}\varphi_3'(h_1)\,w_3^2(h_1)+4\varphi_3(h_1)\,w_2(h_1)\,w_3(h_1)-2\varphi_3(h_1)\,w_3^2(h_1)\Big]$$
$$\geqslant\frac{1}{2}\Big[\varphi_1(h_1)\,w_1^2(h_1)-(\varphi_2(h_1)+2\varphi_3(h_1))\,w_2^2(h_1)$$
$$+\varphi_2(0)\,w_2^2(0)+\left(-\frac{\epsilon}{2}\varphi_3'(h_1)-4\varphi_3(h_1)\right)w_3^2(h_1)\Big] \tag{53}$$

On the other hand we have

$$\varphi_2(0)\,w_2^2(0)=\varphi_2(0)(w_2^2(0)-w_2^2(h_1))+\varphi_2(0)\,w_2^2(h_1)$$
$$w_2^2(0)-w_2^2(h_1)=-2\int_0^{h_1}w_2' w_2$$
$$=2\left(\int_0^{h_1}f_2 w_2+a\int_0^{h_1}w_1 w_2-(a+\lambda)\int_0^{h_1}w_2^2\right)$$
$$\leqslant\frac{1}{\lambda}\|f_2\|_0^2+a\int_0^{h_1}w_1^2$$

Pluging this in (52), we obtain

$$\begin{aligned}
&\int \left[ \frac{a+\lambda}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}\varphi_2(0) - \frac{a}{2}\varphi_2 \right] w_1^2 \\
&\quad + \int \left[ \frac{a+\lambda}{2}\varphi_2 + \frac{1}{2}\varphi_2' - \frac{a}{2}\varphi_1 \right] w_2^2 + \int \left[ \frac{\lambda}{2}\varphi_3 - \frac{\epsilon}{4}\varphi_3'' \right] w_3^2 + \frac{1}{2}\epsilon \int \varphi_3 (w_3')^2 \\
&\quad + \frac{1}{2}\Big[ \varphi_1(h_1)\, w_1^2(h_1) + (\varphi_2(0) - \varphi_2(h_1) - 2\varphi_3(h_1))\, w_2^2(h_1) \\
&\quad - \left( \frac{\epsilon}{2}\varphi_3'(h_1) + 4\varphi_3(h_1) \right) w_3^2(h_1) \Big] \\
&\quad \leqslant \frac{1}{2\lambda}\int \varphi_1 f_1^2 + \frac{1}{2\lambda}\int \varphi_2 f_2^2 + \frac{1}{2\lambda}\varphi_2(0)\int f_2^2 + \frac{1}{2\lambda}\int \varphi_3 f_3^2
\end{aligned} \tag{54}$$

We shall construct $\varphi_1$, $\varphi_2$, and $\varphi_3$ such that they are positive functions, have lower and upper positive bounds independent of $\lambda$ and satisfy

$$\frac{a+\lambda}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}\varphi_2(0) - \frac{a}{2}\varphi_2 > A_1 > 0$$

$$\frac{a+\lambda}{2}\varphi_2 + \frac{1}{2}\varphi_2' - \frac{a}{2}\varphi_1 > A_2 > 0$$

$$\frac{\lambda}{2}\varphi_3 - \frac{\epsilon}{4}\varphi_3'' > A_3 > 0$$

$$\varphi_2(0) - \varphi_2(h_1) - 2\varphi_3(h_1) > A_4 > 0$$

$$-\frac{\epsilon}{2}\varphi_3'(h_1) - 4\varphi_3(h_1) > A_5 > 0$$

where $A_1$, $A_2$, $A_3$, $A_4$, $A_5$ are positive constants independent of $\lambda$. It is then possible to construct $\varphi_1$, $\varphi_2$, and $\varphi_3$ positive functions bounded below and above, and independent of $\lambda$, such that

$$\frac{a+\lambda}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}\varphi_2(0) - \frac{a}{2}\varphi_2 = \frac{\lambda}{2}\varphi_1$$

$$\frac{a+\lambda}{2}\varphi_2 + \frac{1}{2}\varphi_2' - \frac{a}{2}\varphi_1 = \frac{\lambda}{2}\varphi_2$$

$$-\frac{\epsilon}{2}\varphi_3'' = \frac{\epsilon k}{2}\varphi_3$$

and such that all of the above requirements are satisfied and such that $\varphi_2 \leqslant \varphi_2(0)$. The constant $k$ is a positive number determined by the construction of these functions. We conclude then that $||w||_\varphi \leqslant \frac{1}{\lambda-2}||f||_\varphi \ \forall \lambda > 2$, where $||w||_\varphi = \int \varphi_1 w_1^2 + \int \varphi_2 w_2^2 + \int \varphi_3 w_3^2$. The completion of the proof follows from Hille–Yosida theorem.

### 3.2. Asymptotic Analysis of the Coupled Systems

We shall assume that there are nonegative constants $g$ and $h$ such that: $\lim_{t\to\infty} g(t) = g$ and $\lim_{t\to\infty} h(t) = h$. We shall also assume that $ah_1 > 2$ $(a = \frac{1}{\epsilon})$. Then we have the following result about the asymptotic behaviour of the solution of the coupled problem (41)–(46) for large time.

**Theorem 3.2.** Assume that $u_{10}, v_{10} \in L^2[0, h_1]$ and $\rho_{g0} \in L^2[h_1, 1]$. Then the solution of the coupled problem (41)–(46) converges as $t$ tends to $+\infty$ to the solution of the corresponding steady problem.

*Proof.* Without loss of generality, we may assume that $g = g(t) = 0$ and $h = h(t) = 0$. Let $(u_s, v_s, \rho_s)$ denote the solution of the steady problem corresponding to Problem (41)–(46). Such steady solution exists and is unique. Let $\bar{u}_1$, $\bar{v}_1$, and $\bar{\rho}_g$ be defined as follows

$$\begin{aligned} &\bar{u}_1 = u_1 - u_s \quad \text{and} \quad \bar{v}_1 = v_1 - v_s, \quad x \in\ ]0, h_1[, \quad t > 0, \\ &\bar{\rho}_g = \rho_g - \rho_s, \quad x \in\ ]h_1, 1[, \quad t > 0, \end{aligned} \tag{55}$$

where $(u_1, v_1, \rho_g)$ is the solution of the coupled problem (41)–(46). We then have

$$\frac{\partial \bar{u}_1}{\partial t} + \frac{\partial \bar{u}_1}{\partial x} = a(\bar{v}_1 - \bar{u}_1), \qquad x \in\ ]0, h_1[, \quad t > 0, \tag{56}$$

$$\frac{\partial \bar{v}_1}{\partial t} - \frac{\partial \bar{v}_1}{\partial x} = a(\bar{u}_1 - \bar{v}_1), \qquad x \in\ ]0, h_1[, \quad t > 0, \tag{57}$$

$$\bar{u}_1(t, 0) = 0 \qquad \bar{v}_1(t, h_1) = \frac{1}{2}\left(\bar{\rho}_g + \frac{\epsilon}{2}\frac{\partial \bar{\rho}_g}{\partial x}\right)(t, h_1), \qquad t > 0, \tag{58}$$

$$\frac{\partial \bar{\rho}_g}{\partial t} - \frac{\epsilon}{2}\frac{\partial^2 \rho_g}{\partial x^2} = 0, \qquad x \in\ ]h_1, 1[, \quad t > 0, \tag{59}$$

$$\bar{\rho}_g(t, h_1) = \bar{u}_1(t, h_1) + \bar{v}_1(t, h_1) \qquad \bar{\rho}_g(t, 1) = 0, \quad t > 0, \tag{60}$$

$$\bar{u}_1(0, \cdot) = \bar{u}_{10} \qquad \bar{v}_1(0, \cdot) = \bar{v}_{10} \qquad \bar{\rho}_g(0, \cdot) = \bar{\rho}_{g0}. \tag{61}$$

We shall omit the bar sign. Let $\varphi_1$, $\varphi_2$ and $\varphi_3$ be three positive functions independent of $t$ to be precised later. Multiplying Eqs. (56), (57) and

(59) respectively by $\varphi_1 u_1$, $\varphi_2 v_2$ and $\varphi_3 \rho_g$, integrating over $[0, h_1]$ respectively $[h_1, 1]$, using Cauchy–Schwarz inequality, and combining the resulting inequalities, we obtain

$$\frac{d}{dt}\left[\int_0^{h_1} (\varphi_1 u_1^2 + \varphi_2 v_1^2) + \int_{h_1}^1 \varphi_3 \rho_g^2\right] + \int_0^{h_1} (-\varphi_{1x} + a(\varphi_1 - \varphi_2))\, u_1^2$$
$$+ \int_0^{h_1} (\varphi_{2x} + a(\varphi_2 - \varphi_1))\, v_1^2 + \epsilon \int_0^1 \varphi_3 \left|\frac{\partial \rho_g}{\partial x}\right|^2 - \frac{\epsilon}{2}\int_{h_1}^1 \varphi_3'' \rho_g^2$$
$$+ \frac{\epsilon}{2}(\varphi_3' \rho_g^2)_{h_1}^1 - \epsilon \left(\varphi_3 \rho_g \frac{\partial \rho_g}{\partial x}\right)_{h_1}^1 + (\varphi_1 u_1^2 - \varphi_2 v_1^2)_0^{h_1} \leqslant 0 \qquad (62)$$

Using the coupling boundary conditions, we obtain

$$\frac{\epsilon}{2}\left(-\varphi_3'(h_1)\, \rho_g^2(h_1) + 2\varphi_3(h_1)\, \rho_g(h_1)\, \frac{\partial \rho_g}{\partial x}(h_1)\right)$$
$$= -\frac{\epsilon}{2}\varphi_3'(h_1)\, \rho_g^2(h_1) + \varphi_3(h_1)\, \rho_g(h_1)(4 v_1(h_1) - 2\rho_g(h_1))$$
$$\geqslant \left(-\frac{\epsilon}{2}\varphi_3'(h_1) - 4\varphi_3(h_1)\right)\rho_g^2(h_1) - 2\varphi_3(h_1)\, v_1^2(h_1) \qquad (63)$$

Moreover, we have

$$\varphi_2(0)\, v_1^2(0) - 2\varphi_3(h_1)\, v_1^2(h_1) = (\varphi_2(0) - 2\varphi_3(h_1))\, v_1^2(0) + 2\varphi_3(h_1)(v_1^2(0) - v_1^2(h_1))$$
$$v_1^2(0) - v_1^2(h_1) = -2\int_0^{h_1} \frac{\partial v_1}{\partial x}\, v_1$$
$$= -\frac{d}{dt}\int_0^{h_1} v_1^2 - 2a\int_0^{h_1} v_1^2 + 2a\int_0^{h_1} u_1 v_1 \qquad (64)$$

Combining (62), (63) and (64), we obtain

$$\frac{d}{dt}\left[\int_0^{h_1} (\varphi_1 u_1^2 + (\varphi_2 - 2\varphi_3(h_1))\, v_1^2) + \int_{h_1}^1 \varphi_3 \rho_g^2\right]$$
$$+ \int_0^{h_1} (-\varphi_{1x} + a(\varphi_1 - \varphi_2) - 2a\varphi_3(h_1))\, u_1^2$$

$$+\int_0^{h_1} (\varphi_{2x} + a(\varphi_2 - \varphi_1) - 6a\varphi_3(h_1))\, v_1^2$$

$$+\epsilon \int_{h_1}^1 \varphi_3 \left|\frac{\partial \rho_g}{\partial x}\right|^2 - \frac{\epsilon}{2}\int_{h_1}^1 \varphi_3'' \rho_g^2 + \varphi_1(h_1)\, u_1^2(h_1) + (\varphi_2(0) - 2\varphi_3(h_1))\, v_1^2(0)$$

$$+\left(-\frac{\epsilon}{2}\varphi_3'(h_1) - 4\varphi_3(h_1)\right)\rho_g^2(h_1) \leqslant 0 \qquad (65)$$

We shall choose $\varphi_1$, $\varphi_2$ and $\varphi_3$ such that

$$-\varphi_{1x} + a\varphi_1 - a\varphi_2 - 2a\varphi_3(h_1) = k_1 \qquad \text{on} \quad ]0, h_1[$$

$$\varphi_{2x} + a\varphi_2 - a\varphi_1 - 6a\varphi_3(h_1) = k_2 \qquad \text{on} \quad ]0, h_1[$$

$$-\frac{\epsilon}{2}\varphi_3'' = \frac{\epsilon k}{2}\varphi_3 \qquad \text{on} \quad ]h_1, 1[$$

$$\varphi_2(0) - 2\varphi_3(h_1) > 0$$

$$-\frac{\epsilon}{2}\varphi_3'(h_1) - 4\varphi_3(h_1) > 0$$

$$\varphi_2 > 2\varphi_3(h_1)$$

where $k_1$ and $k_2$ are positive constants. It is possible to choose $k_1 > 0$, $k_2 > 0$ and $k > 0$, and construct $\varphi_1$, $\varphi_2$, and $\varphi_3$ positive functions, bounded below and above such that all of the above requirements are satisfied. The conclusion of the proof of the theorem is then a consequence of Gronwall lemma. We then obtain the exponential decay as $t$ goes to $\infty$ of the solution $(u_1, v_1, \rho_g)$.

### 3.3. Convergence Analysis of the Transmission Time Marching Algorithm

In this paragraph, we shall propose an algorithm for the solution of the coupled problem (41)–(46). We then prove that the resulting algorithm converges. As in the previous section we shall assume that there are nonegative constants $g$ and $h$ such that: $\lim_{t\to\infty} g(t) = g$ and $\lim_{t\to\infty} h(t) = h$. We shall also assume that $ah_1 > 2$ ($a = \frac{1}{\epsilon}$).

Applying the transmission time marching algorithm to Problem (41)–(46), we obtain

$$\frac{u_1^{n+1}-u_1^n}{\Delta t}+\frac{du_1^{n+1}}{dx}=a(v_1^{n+1}-u_1^{n+1}) \qquad \text{on} \quad ]0,h_1[, \tag{66}$$

$$\frac{v_1^{n+1}-v_1^n}{\Delta t}-\frac{dv_1^{n+1}}{dx}=a(u_1^{n+1}-v_1^{n+1}) \qquad \text{on} \quad ]0,h_1[, \tag{67}$$

$$u_1^{n+1}(0)=0, \qquad v_1^{n+1}(h_1)=\frac{1}{2}\left(\rho_g^{n+1}+\frac{\epsilon}{2}\frac{\partial\rho_g^{n+1}}{\partial x}\right)(h_1) \tag{68}$$

$$\frac{\rho_g^{n+1}-\rho_g^n}{\Delta t}-\frac{1}{2}\epsilon\frac{d^2\rho_g^{n+1}}{dx^2}=0 \qquad \text{on} \quad ]h_1,1[, \tag{69}$$

$$\rho_g^{n+1}(h_1)=u_1^n(h_1)+v_1^n(h_1), \qquad \rho_g^{n+1}(1)=0, \tag{70}$$

and the initial conditions

$$u_1^0=u_{10}, \qquad v_1^0=v_{10}, \qquad \rho_g^0=\rho_{g0} \tag{71}$$

Here, without loss of generality, we assume that $g=g(t)=0$ and $h=h(t)=0$. The convergence of the algorithm (66)–(71) is stated in the following theorem.

**Theorem 3.3.** The algorithm (66)–(71) converges as $n$ tends to $\infty$.

***Proof.*** Introducing the notations $u_1=u_1^{n+1}$, $v_1=v_1^{n+1}$, $f_1=u_1^n$, $g_1=v_1^n$, $\rho_g=\rho_g^{n+1}$, and $f_g=\rho_g^n$, the algorithm (66)–(71) becomes

$$\begin{cases} bu_1+u_1'=av_1+cf_1 & \text{on} \quad ]0,h_1[, \\ bv_1-v_1'=au_1+cg_1 & \text{on} \quad ]0,h_1[, \\ u_1(0)=0 \qquad v_1(h_1)=\frac{1}{2}\left(\rho_g(h_1)+\frac{\epsilon}{2}\frac{d\rho_g}{dx}(h_1)\right) \end{cases} \tag{72}$$

$$\begin{cases} c\rho_g-\frac{1}{2}\epsilon\frac{d^2\rho_g}{dx^2}=cf_g & \text{on} \quad ]h_1,1[, \\ \rho_g(h_1)=f_1(h_1)+g_1(h_1) \qquad \rho_g(1)=0, \end{cases} \tag{73}$$

with the initial conditions (71). Here, we have used the notation $b=a+\frac{1}{\Delta t}$ and $c=\frac{1}{\Delta t}$. Let $\varphi_1$, $\varphi_2$, and $\varphi_3$ be three positive functions bounded below and above to be precised later. Multiplying the equations in (72) and (73) respectively by $\varphi_1 u_1$, $\varphi_2 v_1$, and $\varphi_3 \rho_g$, integrating respectively over $[0, h_1]$ and $[h_1, 1]$, and using Cauchy–Schwarz inequality, we obtain

$$\int_0^{h_1} \left[ \frac{a+c}{2} \varphi_1 - \frac{1}{2} \varphi_1' - \frac{a}{2} \varphi_2 \right] u_1^2 + \int_0^{h_1} \left[ \frac{a+c}{2} \varphi_2 + \frac{1}{2} \varphi_2' - \frac{a}{2} \varphi_1 \right] v_1^2 + \frac{1}{2} (\varphi_1 u_1^2 - \varphi_2 v_1^2)_0^{h_1} \leqslant \frac{c}{2} \int_0^{h_1} (\varphi_1 f_1^2 + \varphi_2 g_1^2) \tag{74}$$

Integrating by parts, using Cauchy–Schwarz inequality, and the boundary conditions, we obtain

$$\int_{h_1}^1 \left( \frac{c}{2} \varphi_3 - \frac{1}{4} \epsilon \varphi_3'' \right) \rho_g^2 + \frac{\epsilon}{2} \int_{h_1}^1 \varphi_3 (\rho_g')^2 - \frac{1}{4} \epsilon \varphi_3'(h_1)\, \rho_g^2(h_1) + \frac{1}{2} \epsilon \varphi_3(h_1)\, \rho_g'(h_1)\, \rho_g(h_1) \leqslant \frac{c}{2} \int_{h_1}^1 \varphi_3 f_g^2 \tag{75}$$

Assuming that $\varphi_2(h_1) = 0$, the boundary terms become

$$\tfrac{1}{2}(\varphi_1 u_1^2 - \varphi_2 v_1^2)_0^{h_1} + \tfrac{1}{2} \epsilon \varphi_3(h_1)\, \rho_g'(h_1)\, \rho_g(h_1) = \tfrac{1}{2}(\varphi_1(h_1)\, u_1^2(h_1) + \epsilon \varphi_3(h_1)\, \rho_g'(h_1)\, \rho_g(h_1) - \varphi_1(0)\, u_1^2(0) + \varphi_2(0)\, v_1^2(0)) \tag{76}$$

Using the coupling boundary conditions, we obtain

$$\begin{aligned} \epsilon \varphi_3(h_1)\, \rho_g'(h_1)\, \rho_g(h_1) &= 4\varphi_3(h_1)(v_1(h_1)\, \rho_g(h_1) - \tfrac{1}{2} \rho_g^2(h_1)) \\ &= -2\rho_g^2(h_1)\, \varphi_3(h_1) + 4\varphi_3(h_1)\, v_1(h_1)\, \rho_g(h_1) \\ &\geqslant -2\varphi_3(h_1)\, v_1^2(h_1) - 4\varphi_3(h_1)\, \rho_g^2(h_1) \end{aligned} \tag{77}$$

On the other hand we have

$$\varphi_2(0)\, v_1^2(0) = \varphi_2(0)\, v_1^2(h_1) - 2b\varphi_2(0) \int_0^{h_1} v_1^2 + 2a\varphi_2(0) \int_0^{h_1} v_1 u_1 + 2c\varphi_2(0) \int_0^{h_1} v_1 g_1 \tag{78}$$

Hence combining (74), (75), (77), and (78), we obtain

$$\int_0^{h_1} \left[ \frac{a+c}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}\varphi_2 - a\varphi_2(0) \right] u_1^2$$

$$+\int_0^{h_1} \left[ \frac{a+c}{2}\varphi_2 + \frac{1}{2}\varphi_2' - \frac{a}{2}\varphi_1 - (a+c+2b)\,\varphi_2(0) \right] v_1^2$$

$$+\int_{h_1}^1 \left( \frac{c}{2}\varphi_3 - \frac{1}{4}\epsilon\varphi_3'' \right) \rho_g^2 + \frac{\epsilon}{2}\int_{h_1}^1 \varphi_3(\rho_g')^2 + \frac{1}{2}(\varphi_1(h_1)\,u_1^2(h_1) - \varphi_1(0)\,u_1^2(0))$$

$$-\frac{1}{4}\epsilon\varphi_3'(h_1)\,\rho_g^2(h_1) - 4\varphi_3(h_1)\,\rho_g^2(h_1) + (\varphi_2(0) - \varphi_3(h_1))\,v_1^2(h_1)$$

$$\leqslant \frac{c}{2}\int_{h_1}^1 \varphi_2 f_g^2 + \frac{c}{2}\int_0^{h_1} (\varphi_1 f_1^2 + (\varphi_2 + 2\varphi_2(0))\,g_1^2)$$

We shall construct $\varphi_1$, $\varphi_2$, and $\varphi_3$ such that

$$\frac{a+c}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}\varphi_2 - a\varphi_2(0) > \frac{c}{2}\varphi_1$$

$$\frac{b}{2}\varphi_2 + \frac{1}{2}\varphi_2' - (a+c+2b)\,\varphi_2(0) > \frac{c}{2}\varphi_2 + c\varphi_2(0)$$

$$\frac{c}{2}\varphi_3 - \frac{1}{4}\epsilon\varphi_3'' > \frac{c}{2}\varphi_3$$

$$\varphi_2(0) - 2\varphi_3(h_1) > A_1$$

$$\varphi_1(h_1) > B_1$$

where $A_1$ and $B_1$ are positive constants. It is possible to construct $\varphi_1$, $\varphi_2$, and $\varphi_3$ positive functions, bounded below and above, such that all of the above requirements are satisfied. We then conclude that the operator $(u_1^n, v_1^n, \rho_g^n) \to (u_1^{n+1}, v_1^{n+1}, \rho_g^{n+1})$ is a contraction with a constant of contraction $< 1$.

## 4. COUPLING OF KINETIC AND THEIR HYDRODYNAMIC LIMITS: THE MODEL (β)

Let $X = ]0, 1[$ and assume now that the hydrodynamic theory gives a good approximation everywhere except on the boundary. Then we take

$X_1 = ]0, h_1[$ $(0 < h_1 < 1)$ where $h_1$ is small, and $X_2 = X$. Our proposed model consists then of the following physical model: the kinetic model (here the Carleman model) used in $X_1$, which is a small neighborhood of the surface and its hydrodynamic approximation (here the Navier–Stokes like equation) used globally in $X_2$,

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} = a(v_1 - u_1), \qquad x \in ]0, h_1[, \quad t > 0, \tag{79}$$

$$\frac{\partial v_1}{\partial t} - \frac{\partial v_1}{\partial x} = a(u_1 - v_1), \qquad x \in ]0, h_1[, \quad t > 0, \tag{80}$$

$$u_1(t, 0) = g(t) \qquad v_1(t, h_1) = \frac{1}{2}\left( \rho_g + \frac{\epsilon}{2} \frac{\partial \rho_g}{\partial x} \right)(t, h_1), \quad t > 0, \tag{81}$$

$$u_1(0, \cdot) = u_{10} \qquad v_1(0, \cdot) = v_{10}, \quad x \in ]0, h_1[ \tag{82}$$

$$\frac{\partial \rho_g}{\partial t} - \frac{1}{2} \epsilon \frac{\partial^2 \rho_g}{\partial x^2} = 0, \qquad x \in ]0, 1[, \quad t > 0, \tag{83}$$

$$\frac{\partial \rho_g}{\partial x}(., 0) = \frac{2}{\epsilon} v_1(., 0) \qquad \rho_g(t, 1) = h(t), \quad t > 0 \tag{84}$$

$$\rho_g(0, \cdot) = \rho_{g0}, \qquad x \in ]0, 1[ \tag{85}$$

where $u_{10}$, $v_{10}$, and $\rho_{g0}$ are given nonegative given functions. The transmission boundary condition (81) at $x = h_1$ is obtained using the hydrodynamic limit analysis of Section 2, while the transmission condition (84) at $x = 0$ is obtained using the kinetic definition of the flux $\frac{\partial \rho_g}{\partial x}$. The resulting model is called the model $(\beta)$. As for the model $(\alpha)$ it is clear using the analysis of Section 2 that the model $(\beta)$ is rigorously justified. Notice that Problems (79)–(82) and (83)–(85) are only coupled by their boundary conditions. Therefore they can be solved by two independent solution techniques.

### 4.1. Existence Theory

In this paragraph, we shall study the existence and uniqueness of a solution for the coupled problem (79)–(85). We shall work in the Hilbert space

$$H = (L^2[0, h_1])^2 \times (L^2[0, 1]),$$

with the following norm

$$||(w_1, w_2, w_3)|| = (||w_1||^2_{L^2[0, h_1]} + ||w_2||^2_{L^2[0, h_1]} + ||w_3||^2_{L^2[0, 1]})^{\frac{1}{2}}$$

We have the following result.

**Theorem 4.1.** Assume that $(u_{10}, v_{10}, \rho_{g0}) \in H$, then the coupled problem (79)–(85) has a unique strong solution $(u_1, v_1, \rho_g)$.

We shall give the proof of this theorem for the homogeneous boundary conditions: $g(t) = 0$ and $h(t) = 0$. By a standard argument the proof in the nonhomogeneous case can be reduced to the homogeneous case. We proceed as in the previous section.

We introduce an operator $A$ on $H$ as follows

$$A(w_1, w_2, w_3) = \begin{pmatrix} w_1' + a(w_1 - w_2) \\ -w_2' + a(w_2 - w_1) \\ -\frac{1}{2}\epsilon w_3'' \end{pmatrix}, \tag{86}$$

$$D(A) = \left\{ \begin{array}{r} (w_1, w_2, w_3) \in H \mid w_1',\ w_2' \in L^2[0, h_1],\ \text{and}\, w_3'' \in L^2[0, 1] \\ w_1(0) = 0,\ w_2(h_1) = \frac{1}{2} w_3(h_1) + \frac{\epsilon}{4} w_3'(h_1),\ w_3'(0) = \frac{2}{\epsilon} w_2(0), \\ \text{and}\ w_3(1) = 0 \end{array} \right\}.$$

It is clear that $D(A)$ is dense in $H$. We shall use the Hille–Yosida theorem. Let $\lambda$ be a real number and let $f \in X$. We shall study the problem

$$\text{find } w \in D(A) \text{ solution of } Aw + \lambda w = f, \tag{87}$$

which corresponds to finding $w \in D(A)$ such that

$$w_1' + (a + \lambda)\, w_1 - a w_2 = f_1, \tag{88}$$

$$-w_2' + (a + \lambda)\, w_2 - a w_1 = f_2, \tag{89}$$

$$-\tfrac{1}{2}\epsilon w_3'' + \lambda w_3 = f_3, \tag{90}$$

By a density argument we may assume that $f_1$, $f_2$, and $f_3$ are continuous. By elementary methods it is easy to obtain the general solution of System (88)–(90).

We shall now prove that $(w_1, w_2, w_3)$ satisfies

$$\|w\|_\varphi \leqslant \frac{1}{\lambda - 2} \|f\|_\varphi \qquad \forall \lambda > 2 \tag{91}$$

where $\|w\|_\varphi$ is a norm equivalent to the norm $\|\cdot\|$ to be precised later.

Let $\varphi_1$, $\varphi_2$, and $\varphi_3$ be three positive functions (of $x$ only) to be precised later. Multiplying Eqs. (88)–(90) respectively by $\varphi_1 w_1$ , $\varphi_2 w_2$, and $\varphi_3 w_3$, integrating by parts, and using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\int \left[ \frac{a+\lambda}{2}\varphi_1 - \frac{1}{2}\varphi_1' \right] w_1^2 + \frac{1}{2}(\varphi_1 w_1^2)_0^{h1} \\
&\quad -\frac{a}{2}\int \varphi_1 w_2^2 + \int \left[ \frac{a+\lambda}{2}\varphi_2 + \frac{1}{2}\varphi_2' \right] w_2^2 - \frac{1}{2}(\varphi_2 w_2^2)_0^{h_1} \\
&\quad -\frac{a}{2}\int \varphi_2 w_1^2 + \int \left( \frac{\lambda}{2}\varphi_3 - \frac{\epsilon}{4}\varphi_3'' \right) w_3^2 + \frac{1}{2}\epsilon \int \varphi_3 (w_3')^2 + \frac{\epsilon}{4}(\varphi_3' w_3^2)_0^1 - \frac{1}{2}\epsilon(\varphi_3 w_3' w_3)_0^1 \\
&\leqslant \frac{1}{2\lambda}\int \varphi_1 f_1^2 + \frac{1}{2\lambda}\int \varphi_2 f_2^2 + \frac{1}{2\lambda}\int \varphi_3 f_3^2
\end{aligned} \tag{92}$$

On the other hand we have

$$\begin{aligned}
\varphi_2(0)\, w_2^2(0) &= \varphi_2(0)(w_2^2(0) - w_2^2(h_1)) + \varphi_2(0)\, w_2^2(h_1) \\
w_2^2(0) - w_2^2(h_1) &= 2\int_0^{h_1} (-w_2' w_2) \\
&= 2\left( \int_0^{h_1} f_2 w_2 + a\int_0^{h_1} w_1 w_2 - (a+\lambda)\int_0^{h_1} w_2^2 \right) \\
&\leqslant \frac{1}{\lambda}\|f_2\|_0^2 + a\int_0^{h_1} w_1^2
\end{aligned}$$

Pluging this in (92), we obtain

$$\begin{aligned}
&\int \left[ \frac{a+\lambda}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}(\varphi_2(0) - \varphi_3(0)) - \frac{a}{2}\varphi_2 \right] w_1^2 \\
&\quad + \int \left[ \frac{a+\lambda}{2}\varphi_2 + \frac{1}{2}\varphi_2' - \frac{a}{2}\varphi_1 \right] w_2^2 + \int \left[ \frac{\lambda}{2}\varphi_3 - \frac{\epsilon}{4}\varphi_3'' \right] w_3^2 + \frac{1}{2}\epsilon\int \varphi_3 (w_3')^2 \\
&\quad + \frac{1}{2}\Big[ \varphi_1(h_1)\, w_1^2(h_1) + (\varphi_2(0) - \varphi_2(h_1) - \varphi_3(0))\, w_2^2(h_1) \\
&\quad + \left( -\frac{\epsilon}{2}\varphi_3'(0) - \varphi_3(0) \right) w_3^2(0) \Big] \\
&\quad \leqslant \frac{1}{2\lambda}\int \varphi_1 f_1^2 + \frac{1}{2\lambda}\int \varphi_2 f_2^2 + \frac{1}{2\lambda}(\varphi_2(0) - \varphi_3(0))\int f_2^2 + \frac{1}{2\lambda}\int \varphi_3 f_3^2
\end{aligned} \tag{93}$$

We shall construct $\varphi_1$, $\varphi_2$, and $\varphi_3$ such that they are positive functions, have lower and upper positive bounds independent of $\lambda$ and such that

$$\frac{a+\lambda}{2}\varphi_1-\frac{1}{2}\varphi_1'-\frac{a}{2}(\varphi_2(0)-\varphi_3(0))-\frac{a}{2}\varphi_2>A_1>0$$

$$\frac{a+\lambda}{2}\varphi_2+\frac{1}{2}\varphi_2'-\frac{a}{2}\varphi_1>A_2>0$$

$$\frac{\lambda}{2}\varphi_3-\frac{\epsilon}{4}\varphi_3''>A_3>0$$

$$\varphi_2(0)-\varphi_2(h_1)-\varphi_3(0)>A_4>0$$

$$-\frac{\epsilon}{2}\varphi_3'(0)-\varphi_3(0)>A_5>0$$

where $A_1$, $A_2$, $A_3$, $A_4$, and $A_5$ are positive constants independent of $\lambda$. It is possible to construct $\varphi_1$, $\varphi_2$, and $\varphi_3$, positive, independent on $\lambda$, and bounded below and above by positives constants, satisfy

$$\frac{a+\lambda}{2}\varphi_1-\frac{1}{2}\varphi_1'-\frac{a}{2}(\varphi_2(0)-\varphi_3(0))-\frac{a}{2}\varphi_2=\frac{\lambda}{2}\varphi_1$$

$$\frac{a+\lambda}{2}\varphi_2+\frac{1}{2}\varphi_2'-\frac{a}{2}\varphi_1=\frac{\lambda}{2}\varphi_2$$

$$-\frac{\epsilon}{2}\varphi_3''=\frac{\epsilon k}{2}\varphi_3$$

and such that all of the above requirements are satisfied and such that $\varphi_2\leqslant\varphi_2(0)$. Here $k$ is a positive number whose value is determined by the last condition in the set of requirements. We then conclude that: $||w||_\varphi\leqslant\frac{1}{\lambda-2}||f||_\varphi\ \forall\lambda>2$, where $||w||_\varphi=\int\varphi_1w_1^2+\int\varphi_2w_2^2+\int\varphi_3w_3^2$. The proof of the theorem is then a consequence of Hille–Yosida theorem.

## 4.2. Asymptotic Analysis of the Coupled System

As in Section 3.2 we shall assume that there are nonegative constants $g$ and $h$ such that: $\lim_{t\to\infty}g(t)=g$ and $\lim_{t\to\infty}h(t)=h$. We shall also assume that $ah_1>2$ ($a=\frac{1}{\epsilon}$). Then we have the following result about the large time behaviour of the solution of the coupled problem (79)–(85).

**Theorem 4.2.** Assume that $u_{10},v_{10}\in L^2[0,h_1]$ and $\rho_{g0}\in L^2[0,1]$. Then the solution of the coupled problem (79)–(85) converges as $t$ tends to $+\infty$ to the solution of the corresponding steady problem.

*Proof.* As in the previous paragraph, without loss of generality, we may assume that $g = g(t) = 0$ and $h = h(t) = 0$. Let $(u_s, v_s, \rho_s)$ denote the solution of the steady problem corresponding to Problem (79)–(85). Such steady solution exists and is unique. Let $\bar{u}_1$, $\bar{v}_1$, and $\bar{\rho}_g$ be defined as follows

$$\begin{aligned} &\bar{u}_1 = u_1 - u_s \quad \text{and} \quad \bar{v}_1 = v_1 - v_s, \quad x \in ]0, h_1[, \quad t > 0, \\ &\bar{\rho}_g = \rho_g - \rho_s, \quad x \in ]0, 1[, \quad t > 0, \end{aligned} \tag{94}$$

where $(u_1, v_1, \rho_g)$ is the solution of the coupled problem (79)–(85). We then have

$$\frac{\partial \bar{u}_1}{\partial t} + \frac{\partial \bar{u}_1}{\partial x} = a(\bar{v}_1 - \bar{u}_1), \qquad x \in ]0, h_1[, \quad t > 0, \tag{95}$$

$$\frac{\partial \bar{v}_1}{\partial t} - \frac{\partial \bar{v}_1}{\partial x} = a(\bar{u}_1 - \bar{v}_1), \qquad x \in ]0, h_1[, \quad t > 0, \tag{96}$$

$$\bar{u}_1(t, 0) = 0 \qquad \bar{v}_1(t, h_1) = \frac{1}{2}\left(\bar{\rho}_g + \frac{\epsilon}{2}\frac{\partial \bar{\rho}_g}{\partial x}\right)(t, h_1), \quad t > 0, \tag{97}$$

$$\frac{\partial \bar{\rho}_g}{\partial t} - \frac{\epsilon}{2}\frac{\partial^2 \rho_g}{\partial x^2} = 0, \qquad x \in ]0, 1[, \quad t > 0, \tag{98}$$

$$\frac{\partial \bar{\rho}_g}{\partial x}(t, 0) = \frac{2}{\epsilon} v_1(t, 0) \qquad \bar{\rho}_g(t, 1) = 0, \quad t > 0, \tag{99}$$

$$\bar{u}_1(0, \cdot) = \bar{u}_{10} \qquad \bar{v}_1(0, \cdot) = \bar{v}_{10} \qquad \bar{\rho}_g(0, \cdot) = \bar{\rho}_{g0}. \tag{100}$$

We shall omit the bar sign. Let $\varphi_1$, $\varphi_2$ and $\varphi_3$ be three nonegative functions independent of $t$ to be precised later. Multiplying Eqs. (95), (96) and (98) respectively by $\varphi_1 u_1$, $\varphi_2 v_2$ and $\varphi_3 \rho_g$, integrating over $[0, h_1]$ respectively $[0, 1]$, using Cauchy–Schwarz inequality, and combining the resulting inequalities, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\int_0^{h_1} (\varphi_1 u_1^2 + \varphi_2 v_1^2) + \int_0^1 \varphi_3 \rho_g^2\right] + \int_0^{h_1} (-\varphi_{1x} + a(\varphi_1 - \varphi_2))\, u_1^2 \\ &\quad + \int_0^{h_1} (\varphi_{2x} + a(\varphi_2 - \varphi_1))\, v_1^2 + \epsilon \int_0^1 \varphi_3 \left|\frac{\partial \rho_g}{\partial x}\right|^2 - \frac{\epsilon}{2}\int_0^1 \varphi_3'' \rho_g^2 \\ &\quad + \frac{\epsilon}{2}(\varphi_3' \rho_g^2)_0^1 - \epsilon\left(\varphi_3 \rho_g \frac{\partial \rho_g}{\partial x}\right)_0^1 + (\varphi_1 u_1^2 - \varphi_2 v_1^2)_0^{h_1} \leqslant 0 \end{aligned} \tag{101}$$

Using the coupling boundary conditions, we obtain

$$
\begin{aligned}
&\frac{\epsilon}{2}\left(-\varphi_3'(0)\,\rho_g^2(0)+2\varphi_3(0)\,\rho_g(0)\,\frac{\partial\rho_g}{\partial x}(0)\right)\\
&\quad=-\frac{\epsilon}{2}\,\varphi_3'(0)\,\rho_g^2(0)+\varphi_3(0)\,\rho_g(0)\,v_1(0)\\
&\quad\leqslant-\frac{\epsilon}{2}\,\varphi_3'(0)\,\rho_g^2(0)+\varphi_3(0)\,\rho_g^2(0)+\varphi_3(0)\,v_1^2(0)
\end{aligned}
\tag{102}
$$

Combining (102) and (101), and assuming that $\varphi_2(h_1)=0$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\int_0^{h_1}(\varphi_1 u_1^2+\varphi_2 v_1^2)+\int_0^1\varphi_3\rho_g^2\right]+\int_0^{h_1}(-\varphi_{1x}+a(\varphi_1-\varphi_2))\,u_1^2\\
&\quad+\int_0^{h_1}(\varphi_{2x}+a(\varphi_2-\varphi_1))\,v_1^2+\epsilon\int_0^1\varphi_3\left|\frac{\partial\rho_g}{\partial x}\right|^2-\frac{\epsilon}{2}\int_0^1\varphi_3''\rho_g^2\\
&\quad+\varphi_1(h_1)\,u_1^2(h_1)+(\varphi_2(0)-\varphi_3(0))\,v_1^2(0)+\left(-\frac{\epsilon}{2}\,\varphi_3'(0)-\varphi_3(0)\right)\rho_g^2(0)\\
&\leqslant 0
\end{aligned}
\tag{103}
$$

We shall construct $\varphi_1$, $\varphi_2$ and $\varphi_3$ such that

$$
\begin{aligned}
&-\varphi_{1x}+a\varphi_1-a\varphi_2=k_1 \qquad \text{on} \quad ]0,h_1[\\
&\varphi_{2x}+a\varphi_2-a\varphi_1=k_2 \qquad \text{on} \quad ]0,h_1[\\
&-\frac{\epsilon}{2}\,\varphi_3''=\frac{\epsilon k}{2}\,\varphi_3 \qquad \text{on} \quad ]0,1[,\\
&\varphi_2(0)-\varphi_3(0)>0\\
&-\frac{\epsilon}{2}\,\varphi_3'(0)-\varphi_3(0)>0
\end{aligned}
$$

where $k_1$ and $k_2$ are positive constants. It is then possible to choose $k_1>0$, $k_2>0$ and $k>0$, and construct $\varphi_1$, $\varphi_2$, and $\varphi_3$ positive functions, bounded below and above such that all of the above requirements are satisfied. The conclusion of the proof of the theorem is then a consequence of Gronwall lemma. We then obtain the exponential decay as $t$ goes to $\infty$ of the solution $(u_1, v_1, \rho_g)$.

### 4.3. Convergence Analysis of the Transmission Time Marching Algorithm

In this paragraph, we shall propose an algorithm for the solution of the coupled problem (79)–(85). We then prove that the resulting algorithm converges. We shall make the same assumptions under which we obtained the large time behaviour for the coupled system. Namely, we assume that there are nonegative constants $g$ and $h$ such that: $\lim_{t\to\infty} g(t) = g$ and $\lim_{t\to\infty} h(t) = h$, and $ah_1 > 2$ ($a = \frac{1}{\epsilon}$).

Here we will not apply the transmission time marching algorithm directly to the problem (79)–(85). We instead apply this algorithm to an equivalent problem. The problem (79)–(85) is equivalent to the following moments formulation

$$\frac{\partial \rho_l}{\partial t} + \frac{\partial m_l}{\partial x} = 0, \qquad x \in\, ]0, h_1[, \quad t > 0 \tag{104}$$

$$\frac{\partial m_l}{\partial t} + \frac{\partial \rho_l}{\partial x} = -a m_l, \qquad x \in\, ]0, h_1[, \quad t > 0 \tag{105}$$

$$\frac{1}{2}(\rho_l(t,0) + m_l(t,0)) = g(t), \qquad \rho_l(t, h_1) = \rho_g(t, h_1), \quad t > 0, \tag{106}$$

$$\frac{\partial \rho_g}{\partial t} - \frac{1}{2}\epsilon \frac{\partial^2 \rho_g}{\partial x^2} = 0, \qquad x \in\, ]0, 1[, \quad t > 0, \tag{107}$$

$$\frac{\partial \rho_g}{\partial x}(t,0) = -\frac{2}{\epsilon} m_l(t,0), \qquad \rho_g(t,1) = h(t), \quad t > 0, \tag{108}$$

$$\rho_l(0,\cdot) = u_{10} + v_{10}, \qquad m_l(t,\cdot) = u_{10} - v_{10}, \qquad \rho_g(t,\cdot) = \rho_{g0}, \tag{109}$$

where $\rho_l = u_1 + v_1$ and $m_l = u_1 - v_1$. Here, without loss of generality, we assume that $g = g(t) = 0$ and $h = h(t) = 0$. If we apply the transmission time marching algorithm to Problem (104)–(109), we obtain

$$\frac{\rho_l^{n+1} - \rho_l^n}{\Delta t} + \frac{dm_l^{n+1}}{dx} = 0 \qquad \text{on} \quad ]0, h_1[, \tag{110}$$

$$\frac{m_l^{n+1} - m_l^n}{\Delta t} + \frac{d\rho_l^{n+1}}{dx} = -a m_l^{n+1} \qquad \text{on} \quad ]0, h_1[, \tag{111}$$

$$\rho_l^{n+1}(0) + m_l^{n+1}(0) = 0, \qquad \rho_l^{n+1}(h_1) = \rho_g^{n+1}(h_1) \tag{112}$$

$$\frac{\rho_g^{n+1}-\rho_g^n}{\Delta t}-\frac{1}{2}\epsilon\frac{d^2\rho_g^{n+1}}{dx^2}=0 \qquad \text{on} \quad ]0,1[, \tag{113}$$

$$\frac{d\rho_g^{n+1}}{dx}(0)=-\frac{2}{\epsilon}m_l^n(0), \qquad \rho_g(1)=0, \tag{114}$$

$$\rho_l^0=u_{10}+v_{10}, \qquad m_l^0=u_{10}-v_{10}, \qquad \rho_g^0=\rho_{g0}. \tag{115}$$

The convergence of the algorithm (110)–(115) is stated in the following theorem.

**Theorem 4.3.** The algorithm (110)–(115) converges as $n$ tends to $\infty$.

*Remark.* We observe that applying the transmission time marching algorithm to either the original problem (79)–(85) or its equivalent moment formulation (104)–(109) yields the same solution. However, it turns out that the analysis of the algorithm (110)–(115) is technically easier than the one obtained by using a direct application of the transmission time marching algorithm to Problem (79)–(85).

*Proof.* Introducing the notations $c=\frac{1}{\Delta t}$, $b=a+c$, $\rho_l=\rho_l^{n+1}$, $f_l=\rho_l^n$, $m_l=m_l^{n+1}$, $g_l=m_l^n$, $\rho_g=\rho_g^{n+1}$, and $f_g=\rho_g^n$, the algorithm (110)–(115) becomes

$$c\rho_l+m_l'=cf_l \qquad \text{on} \quad ]0,h_1[, \tag{116}$$

$$bm_l+\rho_l'=cg_l \qquad \text{on} \quad ]0,h_1[, \tag{117}$$

$$\rho_l(0)+m_l(0)=0, \qquad \rho_l(h_1)=\rho_g(h_1) \tag{118}$$

$$c\rho_g-\frac{1}{2}\epsilon\frac{d^2\rho_g}{dx^2}=cf_g \qquad \text{on} \quad ]0,1[, \tag{119}$$

$$\rho_g'(0)=-2ag_l(0), \qquad \rho_g(1)=0, \tag{120}$$

Let $\varphi_1$ and $\varphi_2$ be two positive functions to be precised later. Multiplying Eq. (116) by $\varphi_1\rho_l$, and Eq. (117) by $\varphi_1 m_l$, adding the resulting equations, and using Cauchy–Schwarz inequality, we obtain

$$\int_0^{h_1}\left(\frac{c}{2}\varphi_1-\frac{1}{2}\varphi_1'\right)\rho_l^2+\int_0^{h_1}\left(\left(b-\frac{c}{2}\right)\varphi_1-\frac{1}{2}\varphi_1'\right)m_l^2+(m_l\varphi_1\rho_l)_0^{h_1}$$
$$\leqslant\frac{c}{2}\int_0^{h_1}\varphi_1(f_l^2+g_l^2) \tag{121}$$

On the other hand multiplying Eq. (119) by $\varphi_2 \rho_g$, integrating by parts and using Cauchy–Schwarz inequality and the boundary conditions, we obtain

$$\int_0^1 \left(\frac{c}{2}\varphi_2 - \frac{1}{4}\epsilon\varphi_2''\right)\rho_g^2 + \frac{\epsilon}{2}\int_0^1 \varphi_2(\rho_g')^2 - \frac{1}{4}\epsilon\varphi_2'(0)\,\rho_g^2(0) + \frac{1}{2}\epsilon\varphi_2(0)\,\rho_g'(0)\,\rho_g(0)$$
$$\leqslant \frac{c}{2}\int_0^1 \varphi_2 f_g^2 \tag{122}$$

Using the coupling boundary conditions, we get

$$|\tfrac{1}{2}\epsilon\varphi_2(0)\,\rho_g'(0)\,\rho_g(0)| = |\varphi_2(0)\,g_l(0)\,\rho_g(0)| \leqslant \tfrac{1}{2}\varphi_2(0)^2\,g_l^2(0) + \tfrac{1}{2}\,\rho_g^2(0) \tag{123}$$

For the term $\varphi_1(h_1)\,m_l(h_1)\,\rho_l(h_1)$, we first have

$$m_l(h_1) = m_l(0) + \int_0^{h_1} m_l', \qquad \rho_l(h_1) = \rho_l(0) + \int_0^{h_1} \rho_l'$$

Using Cauchy–Schwarz inequality, we then obtain

$$|m_l(h_1)\,\rho_l(h_1)| \leqslant m_l^2(0) + \rho_l^2(0) + h_1\int_0^{h_1}((m_l')^2 + (\rho_l')^2)$$
$$\leqslant m_l^2(0) + \rho_l^2(0) + 2h_1\left[\int_0^{h_1}(c^2(f_l^2 + \rho_l^2) + (b^2 m_l^2 + c^2 g_l^2))\right]$$

Hence using the boundary conditions we have

$$\varphi_1(h_1)\,|m_l(h_1)\,\rho_l(h_1)|$$
$$\leqslant 2\varphi_1(h_1)\,\rho_l^2(0) + 2h_1\varphi_1(h_1)\left[\int_0^{h_1}(c^2(f_l^2 + \rho_l^2) + (b^2 m_l^2 + c^2 g_l^2))\right] \tag{124}$$

Combining (121), (122), (123), and (124), we then obtain

$$\int_0^{h_1}\left(\frac{c}{2}\varphi_1 - \frac{1}{2}\varphi_1' - 2h_1\varphi_1(h_1)\,c^2\right)\rho_l^2 + \int_0^{h_1}\left(\left(b - \frac{c}{2}\right)\varphi_1 - \frac{1}{2}\varphi_1' - 2h_1\varphi_1(h_1)\,b^2\right)m_l^2$$
$$+ \int_0^1\left(\frac{c}{2}\varphi_2 - \frac{1}{4}\epsilon\varphi_2''\right)\rho_g^2 + \frac{\epsilon}{2}\int_0^1 \varphi_2(\rho_g')^2 - \frac{1}{4}\epsilon\varphi_2'(0)\,\rho_g^2(0)$$
$$+ \frac{1}{2}\epsilon\varphi_2(0)\,\rho_g'(0)\,\rho_g(0) + \varphi_1(0)\,\rho_l^2(0)$$

$$\leqslant 2\varphi_1(h_1)\,\rho_l^2(0)+\int_0^{h_1}\left(\frac{c}{2}\varphi_1+2c^2h_1\varphi_1(h_1)\right)f_l^2$$

$$+\int_0^{h_1}\left(\frac{c}{2}\varphi_1+2c^2h_1\varphi_1(h_1)\right)g_l^2+\frac{c}{2}\int_0^1\varphi_2 f_g^2 \tag{125}$$

We shall now construct $\varphi_1$ and $\varphi_2$ such that

$$\frac{c}{2}\varphi_1-\frac{1}{2}\varphi_1'-2h_1\varphi_1(h_1)\,c^2>\frac{c}{2}\varphi_1+2h_1\varphi_1(h_1)\,c^2$$

$$\left(b-\frac{c}{2}\right)\varphi_1-\frac{1}{2}\varphi_1'-2h_1\varphi_1(h_1)\,b^2>\frac{c}{2}\varphi_1+2h_1\varphi_1(h_1)\,c^2$$

$$\varphi_1(0)-2\varphi_1(h_1)>\frac{1}{2}\varphi_2^2(0)$$

$$\frac{c}{2}\varphi_2-\frac{1}{4}\epsilon\varphi_2''>\frac{c}{2}\varphi_2$$

$$-\frac{1}{4}\epsilon\varphi_2'(0)>\frac{1}{2}$$

We first consider the equation

$$-\tfrac{1}{4}\epsilon\varphi_2''=A$$

where $A$ is a positive constant. The solution is given by

$$\varphi_2(x)=\varphi_2(0)+\varphi_2'(0)\,x-\frac{2}{\epsilon}Ax^2 \tag{126}$$

So if we choose for example

$$\varphi_2'(0)=-\frac{3}{\epsilon}\qquad A=\frac{\epsilon}{2}$$

then $\varphi_2(x)=\varphi_2(0)-\frac{3}{\epsilon}x-x^2>c_1>0$ (127)

and if $\varphi_2(0)=\left(\frac{3}{\epsilon}+3\right)$ then $\varphi_2(x)>\frac{3}{\epsilon}(1-x)+(1-x^2)+2>1$

We shall choose $\varphi_1$ in the form

$$\begin{aligned} \varphi_1(x) &= \varphi_1(0) - 2Bx \\ B &= \alpha(b^2+c^2)\, h_1\varphi_1(h_1) \qquad \alpha > 0 \\ \varphi_1(h_1) &= \varphi_1(0) - 2Bh_1 \\ &= \varphi_1(0) - 2\alpha(b^2+c^2)\, h_1^2\varphi_1(h_1) \\ \varphi_1(h_1) &= \frac{1}{1+2\alpha(b^2+c^2)\, h_1^2}\,\varphi_1(0) \end{aligned} \tag{128}$$

Hence we have

$$\begin{aligned} \varphi_1(x) &= \varphi_1(0) - 2Bx \\ &= \varphi_1(0) - 2\alpha(b^2+c^2)\, h_1 \frac{1}{1+2\alpha(b^2+c^2)\, h_1^2}\,\varphi_1(0)\, x \\ &= \varphi_1(0)\left(1 - 2\alpha(b^2+c^2)\, h_1 \frac{1}{1+2\alpha(b^2+c^2)\, h_1^2}\, x\right) \\ &> 0 \qquad \text{for} \quad \varphi_1(0) > 0 \end{aligned} \tag{129}$$

We want $\varphi_1(0)$ to satisfy

$$\varphi_1(0) > 2\varphi_1(h_1) + \tfrac{1}{2}\,\varphi_2^2(0)$$

We then choose $B$ such that

$$B > 2(b^2+c^2)\, h_1\varphi_1(h_1)$$

This condition is satisfied if $\alpha > 2$. Hence we have

$$\varphi_1(0)\left(1 - \frac{2}{1+2\alpha(b^2+c^2)\, h_1^2}\right) > \frac{1}{2}\,\varphi_2^2(0) = \frac{1}{2}\,(3a+3)^2$$

Since $ah_1 > 2$ ($h_1 > 2\epsilon$), we have

$$1 + 2\alpha(b^2+c^2)\, h_1^2 > 1 + 2\alpha$$

Therefore we have

$$1-\frac{2}{1+2\alpha(b^2+c^2)\,h_1^2}>\frac{2\alpha}{1+2\alpha}>0$$

So if we choose

$$\varphi_1(0)>\frac{1+2\alpha(b^2+c^2)\,h_1^2}{2\alpha(b^2+c^2)\,h_1^2-1}\,\frac{1}{2}\,\varphi_2^2(0),$$

all of the requirments imposed upon the construction of $\varphi_1$ and $\varphi_2$ are satisfied. We conclude then that the operator in (125) is contractant with a constant of contraction less than 1. The proof of the theorem is then established.

*Remark.* It is important to notice that the convergence of the numerical algorithm is obtained under the same condition ($ah_1>2$) under which we obtained the large time behaviour of the coupled system (Theorem 4.2). There were no restrictions imposed upon the time step $\Delta t$.

## 5. CONCLUSION

In this paper we have provided a rigorous derivation of the coupling of kinetic and their hydrodynamic limits for a simplified kinetic model. We have proved global convergence of Chapman–Enskog expansion for this model. We studied two types of coupling: the ($\alpha$) and ($\beta$) models. For each model we established the existence theory and the large time behaviour of the solution. We also proposed algorithms for the solution of these models and we established the convergence theory of the resulting algorithms.

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